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Neutrosophic Gompertz Inverse Weibull Distribution: Generalization Properties Simulation with Application on Under-Five Mortality Rates

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ABSTRACT

This work introduces a new continuous statistical distribution namely Neutrosophic Gompertz Inverse Weibull Distribution (NGoIWD), with four neutrosophic parameters. The NGoIWD used the expansion the Inverse Weibull distribution for Neutrosophic logic by merging it with Neutrosophic Gompertz-G family which was found based on T-X method and Neutrosophic logic in direct method that is mean the parameters for this family be as intervals. The study presents serval basic distribution functions for a new generalization with some illustrations, three methods of presenting the parameters were also presented. In addition to performing a Monte Carlo simulation of NGoIWD using three statistical method and six of unconstrained optimization algorithms to determine its efficiency in parameters estimation. Finally to evaluate the efficiency and flexibility of NGoIWD, real data was used, which represented the mortality rate for children under age five, and compared it with six other distributions that proved to be efficient compared to the other distributions.

Keywords: NGoIWD, Neutrosophic Gompertz-G, Nelder-Mead, Entropy, and WLSE.

INTRODUCTION

Since the emergence of statistical distributions and their use in modeling, researchers have focused on the flexibility of these distributions, so many modifications were made to these distributions to be more suitable and more accurate in modeling. One of most famous methods presented in the T-X method, which established new foundations and another perspective for continuous distributions by sitting additional parameters for basic distribution while maintaining the conditions and characteristics for distributions [1]. After that, more than a hundred families of distributions were presented, including: Kw-TG [2], OEHL-G [3], EOF-G [4], GOIE-G [5], MAPW-X [6], LOG [7], NOGEE−G [8], NGLog-X [9], TIHL [10], and , HOE-Φ [11].

Neutrosophic logic (N.L) was first introduced in 1995, which in an advanced case of fuzzy logic because of the addition of another vectors for Truth and Falsity, which represents the value of uncertainty. N.L attracted the attention of many researchers in the fields of topology, mathematical analysis, and other fields of mathematics, but interest in it in the field of mathematical statistics was not directed to it until recently. It appeared in a limited way on basic distributions, and among the distributions presented in this field were: NLD [12], NBL [13], NEIRD [14], NIGD [15], NTLEED [16], and NIPLD [17].

The Neutrosophic Gompertz-G (N.G.o-G) family is used to generate the new distribution, as this family has CDF and pdf functions as follows:

$$F_{NGo}(x_N, r_N, u_N, \psi_N) = 1 - e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{G(x_N,\psi_N)}{1-G(x_N,\psi_N)}}\right)} \tag{1}$$

$$f_{NGo}(x_N, r_N, u_N, \psi_N) = \frac{r_N g(x_N, \psi_N) e^{u_N\frac{G(x_N,\psi_N)}{1-G(x_N,\psi_N)}}}{\left(1 - G(x_N, \psi_N)\right)^2} e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{G(x_N,\psi_N)}{1-G(x_N,\psi_N)}}\right)} \tag{2}$$

Where

x_N: is neutrosophic random variable has form $x_N = x + I_t$, such as x is determine part, and $I_t \in [I_L, I_U]$ is uncertainty part with I_L, and I_U are lower and upper value for it respectively.

$r_N \in [r_L, r_U]$, and $u_N \in [u_L, u_U]$, are neutrosophic parameters for N.G.o-G.

$G(x_N, \psi_N)$, and $g(x_N, \psi_N)$ are NCDF, and Npdf of neutrosophic baseline distribution respectively, with ψ_N neutrosophic parameters.

The gap in previous studies is related to fact that all statistical distribution deal with certain data and other part of distributions as in [12], [13], [14], [15], [16], and [17], don't have enough flexibility to handle the evolution in data contain uncertain values because they transform the basic to neutrosophic distribution (N.D) without adding additional parameters to more flexibility.

Therefore, this work aims to form N.D with additional parameters to add more flexibility to the basic distribution, while evaluation the performance of the proposed distribution through practical application on uncertain data, as well simulating the estimated parameters.

Part.2 including the NGoIWD with some main functions, part.3 including some Generalization Properties of NGoIWD, part.4 including functions of parameters estimation in three methods part.5 including simulation for NGoIWD by nine methods, part.6 including application on real data an comparison, while in finally part.7 including the conclusions for this work.

The Proposed Distribution NGoIWD

Definition 1: Let X_N be neutrosophic random variable has a form $X_N = X + I_t$, such as X is truth part for X_N, and $I_t \in [X_L, X_U]$ is indeterminate part for X_N, then the CDF and pdf of Inverse Weibull distribution from [18], and expansion to N.L, namely (LIW) has form respectively:

$$G(x) = e^{-q_N x_N^{-p_N}} \quad , q_N, p_N, x_N > 0 \tag{3}$$

$$g(x) = q_N p_N x_N^{-(p_N+1)} e^{-q_N x_N^{-p_N}} \quad , q_N, p_N, x_N > 0 \tag{4}$$

Where $q_N \in [q_L, q_U]$, $p_N \in [p_L, p_U]$, are Neutrosophic parameters of Neutrosophic Inverse Weibull, Note that the LIW reduces to classical IW when $X_L = X_U$.

To give the NCDF of NGoIWD, substitution equation 3 into equation 1 in the form:

$$F(x_N) = 1 - e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}}}\right)}, \quad r_N, u_N, q_N, p_N, x_N > 0 \tag{5}$$

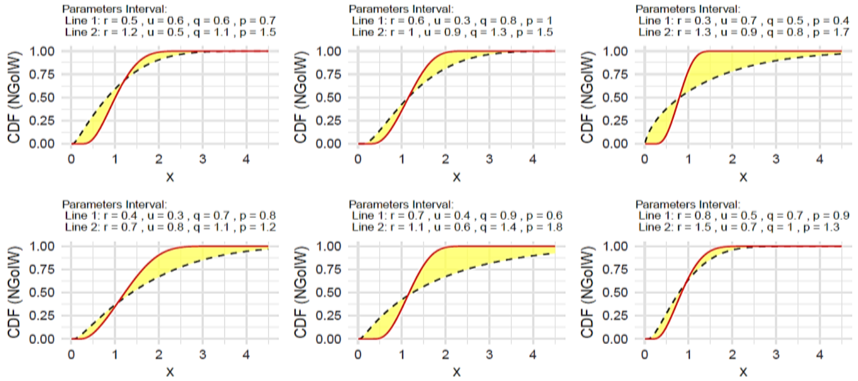


Figure.1 NCDF of NGoIWD with some intervals of parameters

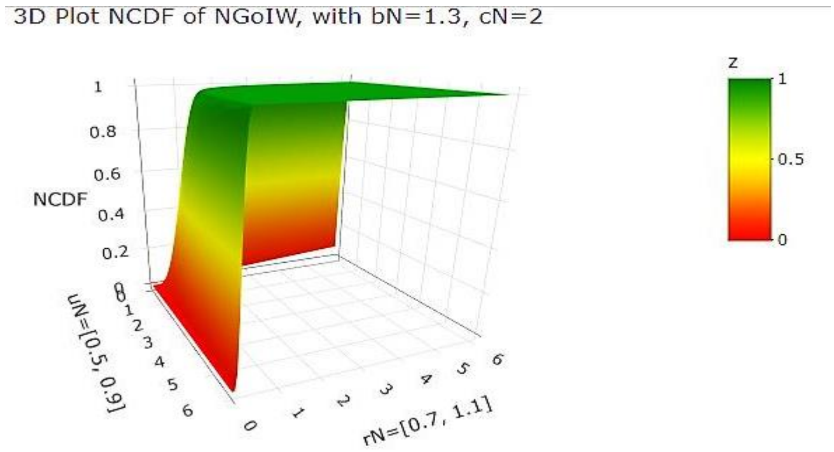


Figure.2 The 3D-plot of NCDF of NGoIWD

The Figure1 show the NCDF of New distribution its noted that all functions range in the interval [0,1], which achieves the property of CDF functions. The red line represents the lower values of neutrosophic random variable, while the black line represents the upper limit of neutrosophic random variable, while the values in color yellow represents the uncertain values. Figure.2 shows the 3-D shape of CDF for NGoIWD, which is consistent with the property shown in Figure1.

By derivative equation 5 we get the Npdf for NGoIWD in form:

$$f(x_N) = \frac{r_N q_N p_N x_N^{-(p_N+1)} e^{-q_N x_N^{-p_N}} e^{u_N \frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}}}}{\left(1-e^{-q_N x_N^{-p_N}}\right)^2} e^{\frac{r_N}{u_N}\left(1-e^{u_N \frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}}}\right)} \qquad (6)$$

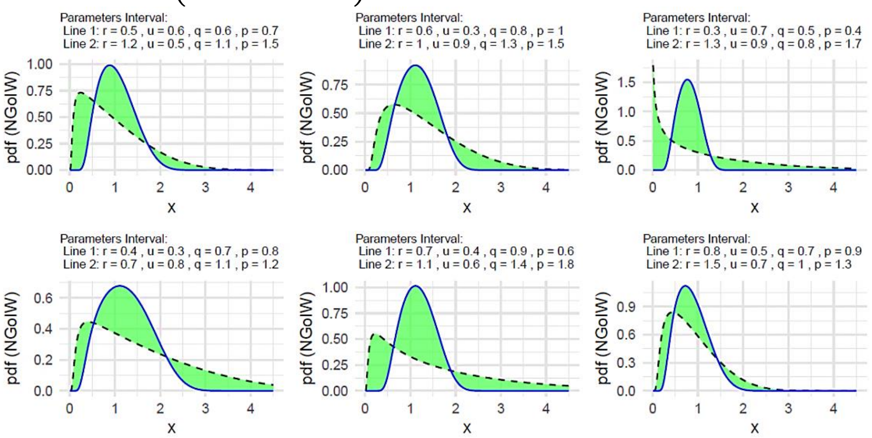


Figure.3 Npdf of NGoIWD with some intervals of parameters

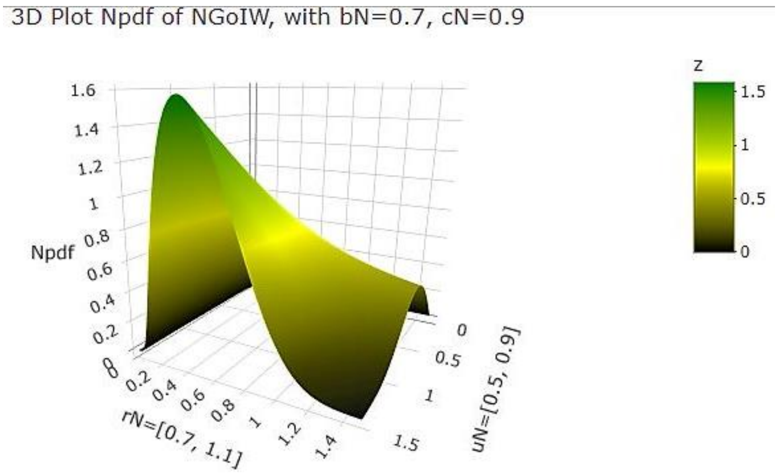


Figure.4 The 3D-plot of Npdf of NGoIWD

Figures 3 and 4 show the flexibility of the distribution with difference or change in parameters intervals.

The survival function allows visual explain of time relationship between failure and continuity using survival curves. These are given by [19], [20]:

$$S(x_N) = e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}}}\right)} \tag{7}$$

As mentioned earlier, the survival function of NGoIWD is shown in figure5, which shows the survival curves of its.

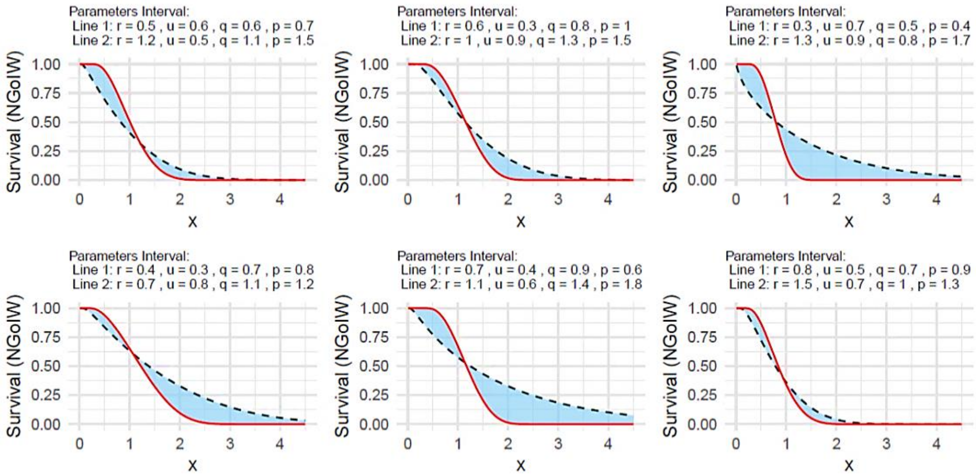


Figure.3 Npdf of NGoIWD with some intervals of parameters

While hazard function provides a means of determining the instantaneous risk at a given time, which helps in understanding the exact timing of failure. It can also calculate the average expected time until failure occurs, which is given as form [21], [22]:

$$h(x_N) = \frac{r_N q_N p_N x_N^{-(p_N+1)} e^{-q_N x_N^{-p_N}} e^{u_N\frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}}}}{\left(1-e^{-q_N x_N^{-p_N}}\right)^2} \tag{8}$$

Generalization Properties of NGoIWD

Useful Expansions

To provide properties and generalizations of the NGoIWD, we must simplify the basic distribution functions due to the difficulty of dealing with them. Using some expansions, as [23], [24], we obtain an expanded NCDF in form:

$$F(x_N) = 1 - Me^{-q_N(k_N+s_N)x_N^{-p_N}} \quad (9)$$

As $M = \sum_{i_N=j_N=k_N=s_N=0}^{\infty} \frac{(-1)^{i_N+j_N+k_N}\Gamma(k_N+s_N)}{i_N!k_N!s_N!\Gamma(k_N)} \binom{i_N}{j_N} r_N^{i_N} u_N^{k_N-i_N} j_N^{k_N}$

While $F^{\delta_N}(x_N)$ has form:

$$F^{\delta_N}(x_N) = \left(1 - e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}}}\right)}\right)^{\delta_N} \quad (10)$$

Can be expansion in form:

$$F^{\delta_N}(x_N) = De^{-q_N(z_N+\beta_N)x_N^{-p_N}} \quad (11)$$

As

$$D = \sum_{l_N=w_N=r_N=z_N=\beta_N=0}^{\infty} \frac{(-1)^{l_N+w_N+r_N+z_N}\Gamma(z_N+\beta_N)}{w_N!\,z_N!\,\beta_N!\,\Gamma(z_N)} \binom{\delta_N}{l_N}\binom{w_N}{r_N} r_N^{w_N} l_N^{w_N} u_N^{z_N-w_N} j_N^{z_N}$$

By same way can expansion Npdf to form:

$$f(x_N) = \mathrm{K}x_N^{-(p_N+1)} e^{-q_N(t_N+v_N+1)x_N^{-p_N}} \quad (12)$$

As

$$\mathrm{K} = \sum_{i_N=j_N=t_N=v_N=0}^{\infty} \frac{(-1)^{i_N+j_N+t_N}\Gamma(t_N+2+v_N)}{i_N!\,u_N^{i_N} t_N!\,v_N!\,\Gamma(t_N+2)} \binom{i_N}{j_N} r_N^{i_N+1} u_N^{t_N}(j_N+1)^{t_N} q_N p_N$$

Finally getting the $f^{\lambda_N}(x_N)$ in form:

$$f^{\lambda_N}(x_N) = \mathrm{W}e^{-q_N(m_N+\varepsilon_N+\lambda_N)x_N^{-p_N}} x_N^{-\lambda_N(p_N+1)} \quad (13)$$

As

$$\mathrm{W} = \sum_{d_N=\gamma_N=m_N=\varepsilon_N=0}^{\infty} \frac{(-1)^{d_N+\gamma_N+m_N}}{d_N!\,u_N^{d_N-m_N}} \binom{d_N}{\gamma_N} r_N^{d_N+\lambda_N} \lambda_N^{d_N} u_N^{m_N}(\gamma_N+\lambda_N)^{m_N} q_N^{\lambda_N} p_N^{\lambda_N}$$

N.Quintile Function

The Neutrosophic Quintile Function is inverse of NCDF expressed as Follows [25]:

$$F(x_N) = s_N$$

Thus its obtained in form:

$$x_N = \left[\frac{-q_N}{\ln\left\{\frac{ln\left[1-\frac{u_N}{r_N}ln(1-s_N)\right]}{u_N+ln\left[1-\frac{u_N}{r_N}ln(1-s_N)\right]}\right\}}\right]^{\frac{1}{p_N}} \quad (14)$$

The following table presents the values of the neutrosophic quintile function for different parameter intervals. To learn more about the values of above function, Table1 was created to show the change in the value of this function when changing the values of parameters intervals and the value of s_N.

Table 1: Quintile function values of NGoIWD for different intervals

s_N	(r_N, u_N, q_N, p_N)				
	[0.5, 0.8],[0.6, 0.9] , [0.7, 1.1],[1.4, 1.7]	**[0.6, 0.9],[0.3, 0.6], [1, 1.4],[1.8, 2.1]**	**[0.3, 0.6],[0.7, 1.0], [1.2, 1.5],[1.2, 1.6]**	**[0.4, 0.7],[0.4, 0.8], [1.3, 1.8],[0.8, 1.7]**	**[0.8, 1.1],[0.5, 0.9] , [1.4, 1.7],[1.7, 2.0]**
0.1	**[0.50968, 0.6649479]**	**[0.69534, 0.7917132]**	**[0.842969,0.86334]**	**[0.91744,0.94790]**	**[0.770805, 0.828741]**
0.2	**[0.6567819, 0.79834]**	**[0.8483596, 0.91875]**	**[1.032068,1.18172]**	**[1.10932,1.27489]**	**[0.9331137, 0.95831]**
0.3	**[0.7821878, 0.90658]**	**[0.9786526, 1.02169]**	**[1.1836584,1.4501]**	**[1.26568,1.5652]**	**[1.061975,1.069771]**
0.4	**[0.898501, 1.003953]**	**[1.10093, 1.114848]**	**[1.31782,1.693755]**	**[1.40655,1.84259]**	**[1.154886,1.197009]**
0.5	**[1.011601, 1.096557]**	**[1.2042553,1.22183]**	**[1.4433156,1.9254]**	**[1.54057,2.11902]**	**[1.24329,1.3220641]**
0.6	**[1.126162, 1.188665]**	**[1.2941056,1.34667]**	**[1.566199,2.15514]**	**[1.67386,2.40497]**	**[1.331488,1.450495]**
0.7	**[1.24766, 1.284887]**	**[1.3889688,1.48192]**	**[1.69257,2.394077]**	**[1.81300,2.71436]**	**[1.423971,1.588843]**
0.8	**[1.38576, 1.3926594]**	**[1.4963977,1.63912]**	**[1.832055,2.66078]**	**[1.96872,3.07282]**	**[1.527998,1.748796]**
0.9	**[1.5313901,1.56638]**	**[1.6363123,1.84993]**	**[2.00906,3.003812]**	**[2.16906,3.55134]**	**[1.662539,1.962021]**

Table 1, shows the value of N.Quintile function for a given range of parameter intervals. The results show how the values of N.Quintiles change as the parameter intervals change, providing an understanding of how changes in parameters affect the probability of NGoIWD.

To find mean for NGoIWD can be substitution $s_N = 0.5$, to get it by form [26]:

$$mean = \left[\frac{-q_N}{\ln\left\{\frac{ln\left[1+0.69315\frac{u_N}{r_N}\right]}{u_N+ln\left[1+0.69315\frac{u_N}{r_N}\right]}\right\}}\right]^{\frac{1}{p_N}} \quad (15)$$

N.Moments

For any neutrosophic random variable x_N, to found neutrosophic moments of order $m-th$ for NGoIWD from [27], and from equation 12 can be derivative at follows:

$$\mu'_{m_N} = \mathrm{K}\int_0^\infty x_N^{m-(p_N+1)} e^{-q_N(t_N+v_N+1)x_N^{-p_N}} dx_N$$

Let $y_N = q_N(t_N+v_N+1)x_N^{-p_N} \Rightarrow x_N = \frac{\sqrt[p_N]{q_N(t_N+v_N+1)}}{y_N^{\frac{1}{p_N}}}$

$$\Rightarrow dx_N = -y_N^{\frac{-1-p_N}{p_N}} \frac{\sqrt[p_N]{q_N(t_N+v_N+1)}}{p_N} dy_N$$

$$\mu'_{m_N} = -\mathrm{K}\int_0^\infty y_N^{\frac{-1-p_N}{p_N}} \frac{\left(q_N(t_N+v_N+1)\right)^{\frac{1}{p_N}}}{p_N} \left(\frac{\left(q_N(t_N+v_N+1)\right)^{\frac{1}{p_N}}}{y_N^{\frac{1}{p_N}}}\right)^{m-(p_N+1)} e^{-y_N} dy_N$$

$$\mu'_{m_N} = \frac{-\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{m}{p_N}-1}}{p_N} \int_0^\infty y_N^{\frac{-m}{p_N}} e^{-y_N} dy_N$$

$$\mu'_{m_N} = \frac{-\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{m}{p_N}-1}}{p_N} \Gamma\left(1-\frac{m}{p_N}\right) \quad (16)$$

Then the first four moments has forms:

$$\mu'_{1_N} = \frac{-\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{1}{p_N}-1}}{p_N} \Gamma\left(1-\frac{1}{p_N}\right) \quad (17)$$

$$\mu'_{2_N} = \frac{-\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{2}{p_N}-1}}{p_N} \Gamma\left(1-\frac{2}{p_N}\right) \quad (18)$$

$$\mu'_{3_N} = \frac{-\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{3}{p_N}-1}}{p_N} \Gamma\left(1-\frac{3}{p_N}\right) \quad (19)$$

$$\mu'_{4_N} = \frac{-\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{4}{p_N}-1}}{p_N} \Gamma\left(1-\frac{4}{p_N}\right) \quad (20)$$

The Neutrosophic variance of the NGoIWD can be founded in formula [27] $\sigma_N^2 = \mu'_{2_N} - {\mu'_{1_N}}^2$ to get:

$$\sigma_N^2 = \frac{-\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{2}{p_N}-1}}{p_N} \Gamma\left(\frac{p_N-2}{p_N}\right) - \left[\frac{\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{1}{p_N}-1}}{p_N} \Gamma\left(\frac{p_N-1}{p_N}\right)\right]^2 \quad (21)$$

Also can founded N.skewness (S_N) and N.kurtosis (K_N) respectively as forms [28], [29]:

$$S_N = \frac{\left(q_N(t_N+v_N+1)\right)^{\frac{1}{p_N}} \Gamma\left(1-\frac{3}{p_N}\right)}{\Gamma\left(1-\frac{2}{p_N}\right)} \quad (22)$$

$$K_N = \frac{(q_N(t_N+v_N+1))^{\frac{2}{p_N}}\Gamma\left(1-\frac{4}{p_N}\right)}{\Gamma\left(1-\frac{2}{p_N}\right)} - 3 \tag{23}$$

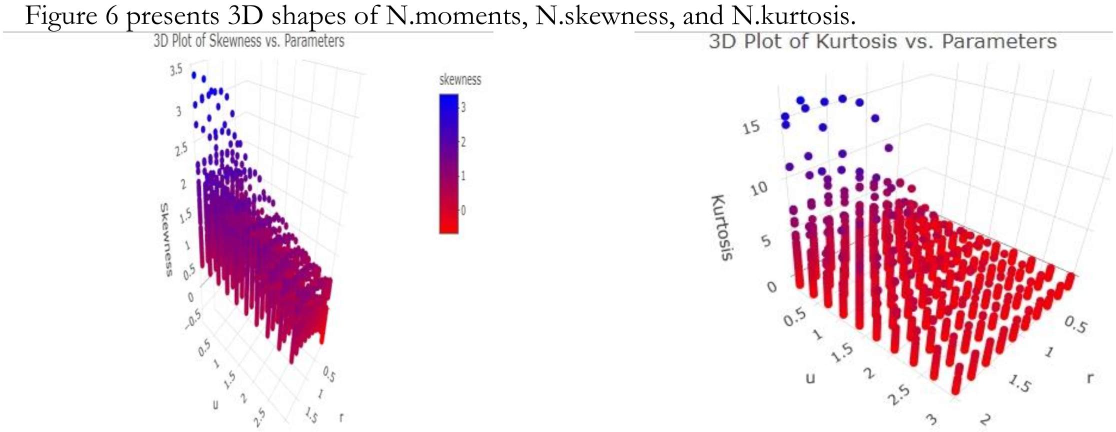
N.moments, N.variance, N.skewness, and N.kurtosis intervals are represented by a set of numbers in Table 2.

Table.2 some intervals of N.moments

r_N	u_N	q_N	p_N	$\acute{\mu}_{1_N}$	$\acute{\mu}_{2_N}$	$\acute{\mu}_{3_N}$	$\acute{\mu}_{4_N}$	σ_N^2	S_N	K_N
[1.2, 1.6]	[1.3,1.7]	[0.6,0.8]	0.1	**[0.12978, 0.2965]**	**[0.63937, 2.77662]**	**[11.9814, 94.8998]**	**[506.391, 7148.5]**	**[0.62253, 2.68871]**	**[20.5112,23.4357]**	**[927.219,1238.74]**
			0.2	**[0.15177, 0.24291]**	**[0.12978, 0.2965]**	**[0.23015, 0.73026]**	**[0.63937, 2.77662]**	**[0.10675, 0.2375]**	**[4.52315,4.92266]**	**[31.5838,37.961]**
		[0.7,0.9]	0.3	**[0.35286, 0.43849]**	**[0.35969, 0.51106]**	**[0.60628, 0.96283]**	**[1.39994, 2.45642]**	**[0.23518, 0.31878]**	**[2.63542,2.8105]**	**[9.40515,10.8206]**
			0.4	**[0.39828, 0.47506]**	**[0.32803, 0.43772]**	**[0.39496, 0.57785]**	**[0.60628, 0.96283]**	**[0.1694, 0.21204]**	**[1.99531,2.10227]**	**[5.02516,5.6345]**
	[1.5,1.9]	[0.8,1]	0.5	**[0.54051, 0.60538]**	**[0.48003, 0.5778]**	**[0.56538, 0.72113]**	**[0.80191, 1.07664]**	**[0.18788, 0.21132]**	**[1.64191,1.69993]**	**[3.22492,3.48]**
			0.6	**[0.57358, 0.63305]**	**[0.48057, 0.56598]**	**[0.50675, 0.62858]**	**[0.62416, 0.8103]**	**[0.15158, 0.16523]**	**[1.47624,1.5211]**	**[2.52954,2.70263]**
		[0.9,1.1]	0.7	**[0.71443, 0.75507]**	**[0.68811, 0.74741]**	**[0.80253, 0.8851]**	**[1.06873, 1.18958]**	**[0.17728,0.1777]**	**[1.36979,1.40597]**	**[2.12949,2.25711]**
			0.8	**[0.73088, 0.76868]**	**[0.68011, 0.73481]**	**[0.74406, 0.81668]**	**[0.91317, 1.01233]**	**[0.14393,0.14593]**	**[1.29656,1.32659]**	**[1.8749,1.97419]**

The table shows that these moments change significantly dependent on parameters, reflecting the flexibility that NGoIWD provides when dealing with uncertain data.

Figure 6 presents 3D shapes of N.moments, N.skewness, and N.kurtosis.



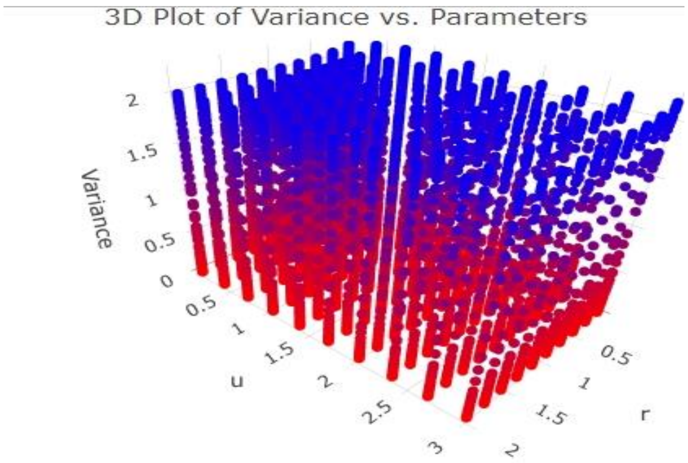


Figure 6. 3.D plot of N.Variance, N.Skewness, and N.Kurtosis

The figure show the relationship between the three moments (N.Variance, N.Skewness, and N.Kurtosis) of NGoIWD in 3.D, taking into account the change in parameters of NGoIWD. The results demonstrate flexibility of NGoIWD when changing the intervals of parameters, which highlights the distribution's compatibility with uncertain data, especially when dealing with data containing different degrees of uncertainty. The N.variance indicates the spread of data, while N.skewness determines its deviation, and N.Kurtosis shows the nature of extremes. It's clear from the figure that these values respond strongly to the change in parameters.

N.Moment Generating Function

The Neutrosophic moment generating function ($N.mgf$) of NGoIWD from equation (16) and by [30] we get:

$$M'_{x_N}(y_N) = \sum_{s=0}^{\infty} \frac{y_N^s}{s!}\left[\frac{-\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{m}{p_N}-1}}{p_N}\Gamma\left(1-\frac{m}{p_N}\right)\right] \tag{24}$$

Characteristic Function

The NGoIWD characteristic function can be found from [31] we get:

$$Q_{x_N}(t_N)_{NGoE} = E(e^{it_N x_N}) = \int_0^{\infty} e^{it_N x_N} f(x_N)\,dx_N \tag{25}$$

By applying the expansion of exponential function and simplifying the above equation based on N. moment equation (16), the characteristic function is obtained as follows:

$$Q_{x_N}(t_N) = \sum_{r=0}^{\infty} \frac{(it_N)^r}{r!}\left[\frac{-\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{m}{p_N}-1}}{p_N}\Gamma\left(1-\frac{m}{p_N}\right)\right] \tag{26}$$

N.Incomplete Moments

The neutrosophic incomplete moments of a random variable x_N may be found using the formula below [25]:

$$\grave{\mu}_{N_r}(y_N) = \int_0^{y_N} x_N^r f(x_N)\,dx_N$$

By replacing NGoIWD from Equation (12) with $f(x_N)$ in the preceding equation, obtain:

$$\grave{\mu}_{N_r}(y_N) = \mathrm{K}\int_0^{y_N} x_N^{r-(p_N+1)} e^{-q_N(t_N+v_N+1)x_N^{-p_N}} dx_N \tag{27}$$

If $x_N = 0 \Longrightarrow s_N = 0$, if $x_N = y_N \Longrightarrow s_N = q_N(t_N + v_N + 1)y_N^{-p_N}$

$$\Longrightarrow y_N = \frac{\left(q_N(t_N + v_N + 1)\right)^{\frac{1}{p_N}}}{s_N^{\frac{1}{p_N}}}$$

By substituting in equation (27), and finding the value of integral, the function will be in form:

$$\grave{\mu}_{N_r}(y_N) = \frac{-\mathrm{K}\left(q_N(t_N+v_N+1)\right)^{\frac{r}{p_N}-1}}{p_N}\Gamma\left(1-\frac{r}{p_N}, \frac{\left(q_N(t_N+v_N+1)\right)^{\frac{1}{p_N}}}{s_N^{\frac{1}{p_N}}}\right) \tag{28}$$

Rényi Entropy

The Rényi entropy of the NGoIWD can be derived from equation (13) and [32], we obtain:

$$I_R(\lambda_N) = \frac{1}{1-\lambda_N}\log\left[\mathrm{W}\int_0^{\infty} x_N^{-\lambda_N(p_N+1)} e^{-q_N(m_N+\varepsilon_N+\lambda_N)x_N^{-p_N}} dx_N\right]$$

Finally we have the Rényi entropy of the NGoIWD in form:

$$I_R(\lambda_N) = \frac{1}{1-\lambda_N}\log\left[\frac{-\mathrm{W}\left(q_N(m_N+\varepsilon_N+\lambda_N)\right)^{\frac{1-\lambda_N(p_N+1)}{p_N}}}{p_N}\Gamma\left(\frac{-1-p_N\lambda_N(p_N+1)}{p_N}\right)\right] \tag{29}$$

Estimation

Maximum likelihood Estimation

The NGoIWD's parameters are determined using maximum likelihood estimation technique. We calculate the log-likelihood for random sample $x_{N_1}, x_{N_2}, \dots, x_{N_m}$. The NGoIWD Npdf is followed [33], [34]:

$$L(\theta_N, x_N) = \prod_{i=1}^{m} f(x_N)$$

$$L(\theta_N, x_{N_i}) = \prod_{i=1}^{m} \frac{r_N q_N p_N x_N^{-(p_N+1)} e^{-q_N x_N^{-p_N}} e^{u_N \frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}}}}{\left(1-e^{-q_N x_N^{-p_N}}\right)^2} e^{\frac{r_N}{u_N}\left(1-e^{u_N \frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}}}\right)}$$

where θ_N is the parameters of distribution, the log-likelihood function L can then be found in the manner described below:

$$\begin{aligned} L = {} & m\log(r_N) + m\log(q_N) + m\log(p_N) - (p_N+1)\sum_{i=1}^{m}\log x_{N_i} - q_N\sum_{i=1}^{m} x_{N_i}^{p_N} \\ & + u_N\sum_{i=1}^{m}\frac{e^{-q_N x_{N_i}^{-p_N}}}{1-e^{-q_N x_{N_i}^{-p_N}}} - 2\sum_{i=1}^{m}\log\left(1-e^{-q_N x_{N_i}^{-p_N}}\right) \\ & - \frac{r_N}{u_N}\sum_{i=1}^{m}\left(1-e^{u_N\frac{e^{-q_N x_{N_i}^{-p_N}}}{1-e^{-q_N x_{N_i}^{-p_N}}}}\right) \end{aligned} \tag{30}$$

Least Square Estimation

The following formula may be used to estimate the parameters using Least Squares Estimation (LSE) approach [28]:

$$\varphi(r_N,u_N,q_N,p_N)=\sum_{i=1}^{m}\left[F\left(x_{N_i}\right)-\frac{1}{n+1}\right]^2$$

$$\varphi(r_N,u_N,q_N,p_N)=\sum_{i=1}^{n}\left[1-e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}}}\right)}-\frac{1}{n+1}\right]^2 \tag{31}$$

Weighted Least Square Estimation

The weighted least squares estimators (WLSE) can be acquired using the formula [28]:

$$\mathrm{W}\,(r_N,u_N,q_N,p_N)=\sum_{i=1}^{m}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{N_i}\right)-\frac{i}{n+1}\right]^2$$

$$\mathrm{W}\,(a_N,b_N,c_N)=\sum_{i=1}^{m}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}}}\right)}-\frac{i}{n+1}\right]^2 \tag{32}$$

To obtain parameter estimates for the three methods above, the partial derivative of four parameters are found and then set equal to zero. Since finding these values is difficult in numerical solutions, computer programs such as the R language are used.

Simulation

The effectiveness of NGoIWD estimator is evaluated by nine methods, three of which are statistical, MLE, LSE, and WLSE, and six other are unconstrained optimization algorithms are, Nelder-Mead, CG, L-BFGS-B, SANN, Newton-Raphson, and Powell. These estimations are used in a monte Carlo simulation exercise conducted using the R programming language. The study includes sample sizes of 50, 100, 150, and 200. Tables 3,4, and 5 present the estimations for these methods with calculation of bias, MSE, and RMSE [35], [36].

Table 3: Monte Carlo simulations conducted for the NGoIWD

$r_N=[0.5,0.8]$, $u_N=[0.3,0.7]$, $q_N=[0.9,1.3]$, $p_N=[0.4,0.8]$					
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**
50	Mean	$\widehat{r_N}$	[0.46668739, 0.6182541]	[0.6054486, 0.9295071]	[0.56499702, 0.8876072]
		$\widehat{u_N}$	[0.1754051, 0.3236452]	[0.36156371, 0.6388827]	[0.292212806, 0.547425]
		$\widehat{q_N}$	[0.79525939, 1.0142120]	[0.94022748, 1.3253124]	[0.91285235, 1.2841601]
		$\widehat{p_N}$	[0.46317615, 1.0335272]	[0.37711064, 0.8113766]	[0.389407034, 0.849375]
	MSE	$\widehat{r_N}$	[0.2686791,0.28538622]	[0.177669, 0.2307012]	[0.07164529, 0.1829713]
		$\widehat{u_N}$	[0.0533528, 0.8509226]	[0.23273454, 0.7466349]	[0.0763242, 0.44296747]
		$\widehat{q_N}$	[0.09051938, 0.2166386]	[0.0448282, 0.0839089]	[0.03333416, 0.0727569]
		$\widehat{p_N}$	[0.01030017, 0.0859805]	[0.01114947, 0.0378672]	[0.00496579, 0.0354836]
	RMSE	$\widehat{r_N}$	[0.5183427,0.53421552]	[0.4215088, 0.4803136]	[0.26766637, 0.4277514]
		$\widehat{u_N}$	[0.2309822, 0.9224547]	[0.48242569, 0.8640803]	[0.27626836, 0.6655579]
		$\widehat{q_N}$	[0.30086439, 0.4654445]	[0.21172670, 0.2896704]	[0.18257644, 0.269735]
		$\widehat{p_N}$	[0.10148974, 0.2932243]	[0.10559107, 0.1945949]	[0.070468415, 0.188371]
	Bias	$\widehat{r_N}$	[0.03331261, 0.1817459]	[0.1054486, 0.1295071]	[0.06499702, 0.0876072]
		$\widehat{u_N}$	[0.1245949, 0.3763548]	[0.06111722,0.06156371]	[0.007787194, 0.152574]
		$\widehat{q_N}$	[0.10474061, 0.285788]	[0.02531243,0.04022748]	[0.01285235, 0.0158398]
		$\widehat{p_N}$	[0.06317615, 0.2335272]	[0.01137663,0.02288936]	[0.01059296, 0.0493751]
100	Mean	$\widehat{r_N}$	[0.46182627, 0.6233728]	[0.55096071, 0.87718518]	[0.53048194, 0.8526529]
		$\widehat{u_N}$	[0.22523849, 0.4171844]	[0.32870632, 0.66729977]	[0.27781516, 0.6139466]
		$\widehat{q_N}$	[0.80222985, 1.0348075]	[0.93150769, 1.30360705]	[0.897025084, 1.286005]
		$\widehat{p_N}$	[0.44584444, 0.9775769]	[0.38624276, 0.81308147]	[0.399328184, 0.832945]
	MSE	$\widehat{r_N}$	[0.13678261, 0.2318716]	[0.04313586, 0.34280687]	[0.04521443, 0.0902951]
		$\widehat{u_N}$	[0.05930992, 0.3138382]	[0.04780708, 0.35448646]	[0.03109868, 0.2656497]

		$\widehat{q_N}$	[0.08680132, 0.1958380]	[0.01861922, 0.06706973]	[0.024944597, 0.037285]
		$\widehat{p_N}$	[0.00682481, 0.0612773]	[0.00268997, 0.02364913]	[0.001900873, 0.018411]
	RMSE	$\widehat{r_N}$	[0.36984133, 0.4815304]	[0.20769174, 0.58549711]	[0.21263685, 0.3004915]
		$\widehat{u_N}$	[0.24353628, 0.5602127]	[0.21864831, 0.59538766]	[0.17634819, 0.5154121]
		$\widehat{q_N}$	[0.29462064, 0.4425359]	[0.13645225, 0.25897825]	[0.15793858, 0.1930932]
		$\widehat{p_N}$	[0.0826124, 0.24754269]	[0.05186492, 0.15378275]	[0.04359900, 0.1356904]
	Bias	$\widehat{r_N}$	[0.03817373, 0.1766272]	[0.05096071, 0.07718518]	[0.03048194, 0.0526529]
		$\widehat{u_N}$	[0.07476151, 0.2828156]	[0.02870632, 0.03270023]	[0.0221848, 0.08605338]
		$\widehat{q_N}$	[0.09777015, 0.2651925]	[0.003607058,0.03150769]	[0.00297491, 0.0139941]
		$\widehat{p_N}$	[0.0458444, 0.17757695]	[0.01308147,0.01375724]	[0.000671815, 0.032945]
150	Mean	$\widehat{r_N}$	[0.48437, 0.6798747]	[0.53580735, 0.85460321]	[0.53494378, 0.80226124]
		$\widehat{u_N}$	[0.25084165, 0.4627168]	[0.32327524, 0.67648691]	[0.27891588, 0.62314157]
		$\widehat{q_N}$	[0.82590556, 1.0758718]	[0.91654314, 1.31519796]	[0.903284323, 1.2709826]
		$\widehat{p_N}$	[0.434643126, 0.941794]	[0.388149795, 0.8170961]	[0.397416375, 0.8320862]
	MSE	$\widehat{r_N}$	[0.1397799, 0.4754326]	[0.03879472, 0.09627254]	[0.02437037, 0.06401544]
		$\widehat{u_N}$	[0.07062188, 0.2523843]	[0.03332547, 0.31107621]	[0.01877535, 0.16126771]
		$\widehat{q_N}$	[0.08653372, 0.1905605]	[0.01739475, 0.03661311]	[0.016432588, 0.0353264]
		$\widehat{p_N}$	[0.004726637, 0.045184]	[0.002004314, 0.0154445]	[0.000991409, 0.0140693]
	RMSE	$\widehat{r_N}$	[0.3738714, 0.6895162]	[0.19696376, 0.31027817]	[0.15611011, 0.25301273]
		$\widehat{u_N}$	[0.26574778, 0.5023786]	[0.18255264, 0.55774207]	[0.13702316, 0.4015815]
		$\widehat{q_N}$	[0.29416615, 0.4365323]	[0.13188917, 0.19134552]	[0.12818965, 0.18795324]
		$\widehat{p_N}$	[0.06875054, 0.2125658]	[0.044769568, 0.1242761]	[0.031486652, 0.1186142]
	Bias	$\widehat{r_N}$	[0.01563, 0.1201253]	[0.03580735, 0.05460321]	[0.002261242,0.03494378]
		$\widehat{u_N}$	[0.04915835, 0.2372832]	[0.02327524, 0.02351309]	[0.02108412, 0.07685843]
		$\widehat{q_N}$	[0.07409444, 0.2241282]	[0.01519796,0.01654314]	[0.003284323, 0.0290173]
		$\widehat{p_N}$	[0.03464312, 0.1417946]	[0.011850205, 0.0170961]	[0.002583625, 0.0320862]
200	Mean	$\widehat{r_N}$	[0.4814127, 0.73798983]	[0.51167894, 0.84080386]	[0.50997524, 0.82970066]
		$\widehat{u_N}$	[0.2808029, 0.5679102]	[0.32862468, 0.66092732]	[0.293564672, 0.6145929]
		$\widehat{q_N}$	[0.82727612, 1.1340021]	[0.901685751, 1.3127085]	[0.8942528, 1.29187967]
		$\widehat{p_N}$	[0.42989117, 0.910865]	[0.39044437, 0.81794223]	[0.400175703, 0.8303868]
	MSE	$\widehat{r_N}$	[0.13401156, 0.625065]	[0.02154909, 0.05663662]	[0.01907696, 0.07932730]
		$\widehat{u_N}$	[0.11258483, 0.4234627]	[0.02835989, 0.22192503]	[0.015847092, 0.1105292]
		$\widehat{q_N}$	[0.0873481, 0.1977034]	[0.01345587, 0.02406205]	[0.013563347, 0.0297300]
		$\widehat{p_N}$	[0.00474767, 0.0377209]	[0.001689513, 0.0108983]	[0.001138321, 0.0095087]
	RMSE	$\widehat{r_N}$	[0.36607589, 0.7906105]	[0.14679607, 0.2379845]	[0.13811938, 0.28165102]
		$\widehat{u_N}$	[0.33553663, 0.6507401]	[0.16840394, 0.4710892]	[0.125885235, 0.3324593]
		$\widehat{q_N}$	[0.29554712, 0.4446385]	[0.115999477, 0.1551194]	[0.1164617, 0.172424103]
		$\widehat{p_N}$	[0.06890335, 0.1942188]	[0.04110368, 0.10439509]	[0.033739019, 0.0975126]
	Bias	$\widehat{r_N}$	[0.01858728, 0.0620101]	[0.01167894, 0.04080386]	[0.00997524, 0.02970066]
		$\widehat{u_N}$	[0.01919703, 0.1320898]	[0.02862468, 0.03907268]	[0.00643532, 0.08540708]
		$\widehat{q_N}$	[0.07272388, 0.1659979]	[0.001685751, 0.0127085]	[0.00574719, 0.00812032]
		$\widehat{p_N}$	[0.0298911, 0.11086507]	[0.00955562, 0.01794223]	[0.000175703, 0.0303868]

Table.4 Monte Carlo simulations conducted for the NGoIWD

$r_N = [0.5, 0.8]$, $u_N = [0.3, 0.7]$, $b_N = [0.9, 1.3]$, $c_N = [0.4, 0.8]$					
N	**Est.**	**Ess. Par.**	**Nelder-Mead**	**CG**	**L-BFGS-B**
50	Mean	$\widehat{r_N}$	[1.355002, 4.975848]	[0.9313228, 1.106194]	[1.23887, 11.98065]
		$\widehat{u_N}$	[0.3399777, 2.43233]	[0.5088168,0.7712783]	[0.3002779, 4.455575]
		$\widehat{q_N}$	[1.636063, 2.912994]	[1.24312, 1.554482]	[1.562448, 3.75069]
		$\widehat{p_N}$	[0.3583603, 0.5502364]	[0.3184571, 0.8508335]	[0.3684675, 0.4487061]
	Bias	$\widehat{r_N}$	[0.8550022, 4.175848]	[0.306194,0.4313228]	[0.7388701, 11.18065]
		$\widehat{u_N}$	[0.03997773, 1.73233]	[0.1911832,0.4712783]	[0.0002779252, 3.7555]
		$\widehat{q_N}$	[0.7360628, 1.612994]	[0.2544825,0.3431199]	[0.6624476, 2.45069]
		$\widehat{p_N}$	[0.04163965, 0.2497636]	[0.05083348,0.0815429]	[0.0315325, 0.3512939]
	MSE	$\widehat{r_N}$	[0.7310287, 17.43771]	[0.09375477,0.1860394]	[0.545929, 125.007]
		$\widehat{u_N}$	[0.00159821, 3.000968]	[0.03655101,0.2221032]	[7.724241e-08, 14.1043]
		$\widehat{q_N}$	[0.5417884, 2.601749]	[0.06476134,0.1177312]	[0.4388369, 6.00588]
		$\widehat{p_N}$	[0.00173386, 0.0623818]	[0.002584043,0.006649245]	[0.00099429, 0.123407]
	RMSE	$\widehat{r_N}$	[0.8550022, 4.175848]	[0.306194,0.4313228]	[0.7388701, 11.18065]
		$\widehat{u_N}$	[0.03997773, 1.73233]	[0.1911832,0.4712783]	[0.000277925, 3.75557]
		$\widehat{q_N}$	[0.7360628, 1.612994]	[0.2544825,0.3431199]	[0.6624476, 2.45069]
		$\widehat{p_N}$	[0.2497636,0.04163965]	[0.05083348,0.0815429]	[0.0315325, 0.3512939]
100	Mean	$\widehat{r_N}$	[1.163651, 1.185493]	[0.4645801, 1.050686]	[0.4417431, 1.101812]
		$\widehat{u_N}$	[0.6047439,1.438699]	[0.4142403, 0.5374379]	[0.3792404, 0.542719]
		$\widehat{q_N}$	[1.620075, 1.645129]	[0.9261985, 1.542371]	[0.8937656, 1.584516]
		$\widehat{p_N}$	[0.2958123, 0.8180557]	[0.4016579, 0.8432044]	[0.409223, 0.8389269]
	Bias	$\widehat{r_N}$	[0.3854929,0.6636512]	[0.03541986, 0.2506863]	[0.0582568, 0.301812]

		$\widehat{u_N}$	[0.09525606,1.138699]	[0.1142403, 0.1625621]	[0.0792404, 0.157280]
		$\widehat{q_N}$	[0.3451291,0.7200753]	[0.0261985, 0.2423713]	[0.0062344, 0.28451]
		$\widehat{p_N}$	[0.01805571,0.1041877]	[0.001657918, 0.04320436]	[0.0092234, 0.038926]
	MSE	$\widehat{r_N}$	[0.1486048,0.440433]	[0.001254567, 0.06284364]	[0.0033938, 0.091090]
		$\widehat{u_N}$	[0.009073717,1.296635]	[0.01305085, 0.02642643]	[0.0062790, 0.024737]
		$\widehat{q_N}$	[0.1191141,0.5185084]	[0.0006863612, 0.0587438]	[3.886786e-05, 0.0809]
		$\widehat{p_N}$	[0.0003260086,0.01085507]	[2.748693e-06, 0.0018666]	[8.5072e-05, 0.00151]
	RMSE	$\widehat{r_N}$	[0.3854929,0.663651]	[0.03541986, 0.2506863]	[0.05825688, 0.30181]
		$\widehat{u_N}$	[0.09525606,1.138699]	[0.1142403, 0.1625621]	[0.07924044, 0.15728]
		$\widehat{q_N}$	[0.3451291,0.7200753]	[0.0261985, 0.2423713]	[0.0062344, 0.28451]
		$\widehat{p_N}$	[0.01805571,0.1041877]	[0.001657918, 0.04320436]	[0.0092234, 0.038926]
150	Mean	$\widehat{r_N}$	[0.1469031, 0.5685071]	[0.7925428,0.9846514]	[0.1512468, 0.4267646]
		$\widehat{u_N}$	[0.551405,2.21236]	[0.7825375, 0.8948623]	[0.3454505,2.32756]
		$\widehat{q_N}$	[0.6592832, 1.057897]	[1.227886, 1.304066]	[0.6804332, 0.8651825]
		$\widehat{p_N}$	[0.1506949, 0.850543]	[0.3052497, 0.7570574]	[0.1485408, 0.9394217]
	Bias	$\widehat{r_N}$	[0.2314929,0.3530969]	[0.007457163,0.4846514]	[0.3487532, 0.3732354]
		$\widehat{u_N}$	[0.148595,1.91236]	[0.1948623,0.4825375]	[0.3545495,2.02756]
		$\widehat{q_N}$	[0.2407168, 0.2421034]	[0.004065934,0.3278861]	[0.2195668, 0.4348175]
		$\widehat{p_N}$	[0.05054298,0.2493051]	[0.04294262,0.09475025]	[0.1394217,0.2514592]
	MSE	$\widehat{r_N}$	[0.05358894,0.1246774]	[,5.560928e-050.234887]	[0.1216288, 0.1393047]
		$\widehat{u_N}$	[0.02208047,3.657121]	[0.03797132,0.2328424]	[0.1257053,4.111001]
		$\widehat{q_N}$	[0.05794458, 0.05861407]	[1.653182e-05,0.1075093]	[0.04820959, 0.1890662]
		$\widehat{p_N}$	[0.002554593,0.06215305]	[0.001844069,0.008977611]	[0.01943842,0.06323171]
	RMSE	$\widehat{r_N}$	[0.2314929,0.3530969]	[0.007457163,0.4846514]	[0.3487532, 0.3732354]
		$\widehat{u_N}$	[0.148595,1.91236]	[0.1948623,0.4825375]	[0.3545495,2.02756]
		$\widehat{q_N}$	[0.2407168, 0.2421034]	[0.004065934,0.3278861]	[0.2195668, 0.4348175]
		$\widehat{p_N}$	[0.05054298,0.2493051]	[0.04294262,0.09475025]	[0.1394217,0.2514592]
200	Mean	$\widehat{r_N}$	[0.5353348, 0.9027327]	[0.848342,0.9613756]	[0.85304,1.150592]
		$\widehat{u_N}$	[0.9209243,2.228872]	[0.7718807, 0.851686]	[0.6264095, 0.84799]
		$\widehat{q_N}$	[1.035185, 1.392548]	[1.267691, 1.34346]	[1.34747,1.439355]
		$\widehat{p_N}$	[0.1982715, 0.7504132]	[0.3126908, 0.76517]	[0.3274602, 0.76630]
	Bias	$\widehat{r_N}$	[0.03533478, 0.1027327]	[0.048342,0.4613756]	[0.053040,0.6505924]
		$\widehat{u_N}$	[0.2209243,1.928872]	[0.15168,0.4718807]	[0.14799,0.3264095]
		$\widehat{q_N}$	[0.09254823,0.1351846]	[0.04346,0.3676906]	[0.04747,0.5393555]
		$\widehat{p_N}$	[0.04958685,0.2017285]	[0.034824,0.08730916]	[0.03369,0.07253984]
	MSE	$\widehat{r_N}$	[0.001248547, 0.010554]	[0.002337,0.2128675]	[0.002813,0.4232704]
		$\widehat{u_N}$	[0.048807,3.720549]	[0.023008,0.2226714]	[0.021902,0.1065432]
		$\widehat{q_N}$	[0.008565,0.01827489]	[0.001889,0.1351964]	[0.002253,0.2909043]
		$\widehat{p_N}$	[0.002458,0.04069438]	[0.0012127,0.007622889]	[0.001135,0.005262028]
	RMSE	$\widehat{r_N}$	[0.03533478, 0.10273]	[0.048342,0.4613756]	[,0.053040.6505924]
		$\widehat{u_N}$	[0.22092,1.928872]	[0.151686,0.4718807]	[0.147994,0.3264095]
		$\widehat{q_N}$	[0.09254,0.1351846]	[0.043463,0.3676906]	[0.04747,0.5393555]
		$\widehat{p_N}$	[0.04958,0.2017285]	[0.03482,0.08730916]	[0.03369,0.07253984]

Table 5: Monte Carlo simulations conducted for the NGoIWD

$r_N = [0.5, 0.8]$, $u_N = [0.3, 0.7]$, $b_N = [0.9, 1.3]$, $c_N = [0.4, 0.8]$					
N	Est.	Ess. Par.	SANN	Newton-Raphson	Powell
50	Mean	$\widehat{r_N}$	[3.84134, 7.12250]	[1.044982, 10.24967]	[0.9313228, 1.10619]
		$\widehat{u_N}$	[1.299868, 3.29770]	[0.219681, 4.58899]	[0.50881,0.7712783]
		$\widehat{q_N}$	[2.497581, 3.25205]	[1.427332, 3.58620]	[1.24312, 1.55448]
		$\widehat{p_N}$	[0.2542516, 0.50052]	[0.3902317, 0.45385]	[0.3184571, 0.85083]
	Bias	$\widehat{r_N}$	[3.34134, 6.32250]	[0.544981, 9.44967]	[0.30619,0.4313228]
		$\widehat{u_N}$	[0.9998678, 2.59770]	[0.08031812, 3.88899]	[0.19118,0.4712783]
		$\widehat{q_N}$	[1.59758, 1.952051]	[0.5273319, 2.28620]	[0.25448,0.3431199]
		$\widehat{p_N}$	[0.1457484, 0.29947]	[0.0097682, 0.346144]	[0.05083,0.0815429]
	MSE	$\widehat{r_N}$	[11.16455, 39.97402]	[0.297005, 89.29635]	[0.09375,0.1860394]
		$\widehat{u_N}$	[0.999735, 6.74807]	[0.0064510, 15.12426]	[0.03655,0.2221032]
		$\widehat{q_N}$	[2.552267, 3.81050]	[0.2780789, 5.22675]	[0.06476,0.1177312]
		$\widehat{p_N}$	[0.02124259, 0.089683]	[9.541895e-05, 0.119816]	[0.006649245, 0.002584]
	RMSE	$\widehat{r_N}$	[3.34134, 6.32250]	[0.544981, 9.449675]	[0.30619,0.4313228]
		$\widehat{u_N}$	[0.9998678, 2.59770]	[0.0803181, 3.88899]	[0.19118,0.4712783]

		$\widehat{q_N}$	[1.59758, 1.952051]	[0.5273319, 2.28620]	[0.25448,0.3431199]
		$\widehat{p_N}$	[0.1457484, 0.299471]	[0.009768263, 0.346144]	[0.050833,0.0815429]
100	Mean	$\widehat{r_N}$	[0.498638, 2.31765]	[0.714961,1.486446]	[0.8130915, 1.05068]
		$\widehat{u_N}$	[0.6356553, 1.51714]	[0.26378,0.7311849]	[0.53743,0.8118429]
		$\widehat{q_N}$	[0.972445, 2.21647]	[0.1632088, 1.24524]	[1.338199, 1.54237]
		$\widehat{p_N}$	[0.3559983, 0.64125]	[0.2696286, 0.973783]	[0.3464501, 0.84320]
	Bias	$\widehat{r_N}$	[0.001361987, 1.51765]	[0.085038,0.9864462]	[0.25068,0.3130915]
		$\widehat{u_N}$	[0.3356553, 0.81714]	[0.4311849, 0.43621]	[0.16256,0.5118429]
		$\widehat{q_N}$	[0.0724450, 0.91647]	[0.05475,0.7367912]	[0.24237,0.4381988]
		$\widehat{p_N}$	[0.04400169, 0.15874]	[0.17378,0.6696286]	[0.04320,0.05354991]
	MSE	$\widehat{r_N}$	[1.855009e-06, 2.30326]	[0.007231,0.973076]	[0.062843,0.09802629]
		$\widehat{u_N}$	[0.1126645, 0.66773]	[0.1859204, 0.19028]	[0.02642,0.2619832]
		$\widehat{q_N}$	[0.005248284, 0.8399]	[0.002998,0.5428613]	[0.058743,0.1920182]
		$\widehat{p_N}$	[0.0019361, 0.02520]	[0.030200,0.4484024]	[0.001866,0.002867593]
	RMSE	$\widehat{r_N}$	[0.001361987, 1.51765]	[0.085038,0.9864462]	[0.250686,0.3130915]
		$\widehat{u_N}$	[0.3356553, 0.81714]	[0.4311849, 0.43621]	[0.16256,0.5118429]
		$\widehat{q_N}$	[0.07244504, 0.91647]	[0.054758,0.7367912]	[0.24237,0.4381988]
		$\widehat{p_N}$	[0.04400169, 0.15874]	[0.17378,0.6696286]	[0.043204,0.05354991]
150	Mean	$\widehat{r_N}$	[1.39865,10.12938]	[0.42384,1.596374]	[0.79254,0.9846514]
		$\widehat{u_N}$	[1.70641,3.672564]	[0.34165,0.6543639]	[0.7825375, 0.89486]
		$\widehat{q_N}$	[1.75430,3.721477]	[0.002944669, 0.86089]	[1.227886, 1.30406]
		$\widehat{p_N}$	[0.2566957, 0.63320]	[0.5908346, 0.94147]	[0.3052497, 0.75705]
	Bias	$\widehat{r_N}$	[0.59865,9.629377]	[0.376155,1.096374]	[0.00745,0.4846514]
		$\widehat{u_N}$	[1.00641,3.972564]	[0.3543639, 0.35834]	[0.194862,0.4825375]
		$\widehat{q_N}$	[0.454303,2.821477]	[0.43910,0.8970553]	[0.004065,0.3278861]
		$\widehat{p_N}$	[0.1433043, 0.16679]	[0.14147,0.9908346]	[0.042942,0.09475025]
	MSE	$\widehat{r_N}$	[0.35839,92.72489]	[0.14149,1.202036]	[5.560928e-05,0.234887]
		$\widehat{u_N}$	[1.01287,15.78126]	[0.1255738, 0.12841]	[0.037971,0.2328424]
		$\widehat{q_N}$	[0.20639,7.960733]	[0.19281,0.8047083]	[1.65318e-05 ,0.1075093]
		$\widehat{p_N}$	[0.02053612, 0.0278217]	[0.02001,0.9817531]	[0.001844,0.008977611]
	RMSE	$\widehat{r_N}$	[0.59865,9.629377]	[0.376155,1.096374]	[0.007457,0.4846514]
		$\widehat{u_N}$	[1.00641,3.972564]	[0.3543639, 0.35834]	[0.19486,0.4825375]
		$\widehat{q_N}$	[0.45430,2.821477]	[0.43910,0.8970553]	[0.00406,0.3278861]
		$\widehat{p_N}$	[0.1433043, 0.166798]	[0.14147,0.9908346]	[0.042942,0.09475025]
200	Mean	$\widehat{r_N}$	[3.42605,12.56142]	[0.56357,1.595847]	[0.848342,0.9613756]
		$\widehat{u_N}$	[0.9804483, 4.37389]	[0.44375,0.6533407]	[0.7718807, 0.85168]
		$\widehat{q_N}$	[2.55036,3.793849]	[0.004949262, 1.03876]	[1.267691, 1.34346]
		$\widehat{p_N}$	[0.2153022, 0.48186]	[0.5920641, 0.891915]	[0.3126908, 0.76517]
	Bias	$\widehat{r_N}$	[2.62605,12.06142]	[0.23642,1.095847]	[0.048342,0.4613756]
		$\widehat{u_N}$	[1.280448, 3.67389]	[0.25624,0.3533407]	[0.15168,0.4718807]
		$\widehat{q_N}$	[1.25036,2.893849]	[0.26123,0.8950507]	[0.043463,0.3676906]
		$\widehat{p_N}$	[0.1846978, 0.31813]	[0.09191,0.9920641]	[0.03482,0.08730916]
	MSE	$\widehat{r_N}$	[6.89614,145.4778]	[0.055894,1.20088]	[0.002337,0.2128675]
		$\widehat{u_N}$	[1.639548, 13.4974]	[0.065659,0.1248497]	[0.023008,0.2226714]
		$\widehat{q_N}$	[1.5634,8.37436]	[0.068244,0.8011158]	[0.001889,0.1351964]
		$\widehat{p_N}$	[0.03411328, 0.10120]	[0.008448,0.9841912]	[0.0012127,0.007622889]
	RMSE	$\widehat{r_N}$	[2.6260,12.06142]	[0.23642,1.095847]	[0.4613756,0.048342]
		$\widehat{u_N}$	[1.280448, 3.6738]	[0.25624,0.3533407]	[0.043463,0.4718807]
		$\widehat{q_N}$	[1.2503,2.893849]	[0.26123,0.8950507]	[0.034824,0.3676906]
		$\widehat{p_N}$	[0.1846978, 0.31813]	[0.09191,0.9920641]	[0.08730916,0.151686]

To analyze Tables 3, 4, and 5 more carefully, one can focus on the following:

- Table 3, analyzes simulation results using traditional statistical estimation (MLE, LSE, WLSE), MLE method showed the lowest bias, and lowest MSE compared to other methods, especially when large sample (n=150 or 200). For distribution parameters, the performance was stable and mostly accurate for mean and variance parameters. At sample size 200, the performance was most stable, with fewer deviations in estimate, showing that the accuracy of the MLE improves with increasing sample size.
- Table 4, shows simulation results for a set of unconstrained optimization algorithms such as (Nelder-Mead, CG, and L-BFGS-B). CG analysis was most effective overall, providing results with less bias and skewness, and low MSE at different sample size, Nelder-Mead also showed good performance but was less stable in some scenarios. Similarly to table 3, performance improved with increasing sample size, with CG performing better at large sample sizes (n=150 or 200).

- Table 5, shows simulation results using advanced estimation such as (SANN, Powell, and Newton-Raphson). Newton-Raphson analysis clearly outperformed other methods such as Powell and SANN in providing accurate estimates of NGoIWD parameters. Results were more stable and less skewed, with lowest MSE in most scenarios. A sample size (n=200) was optimal for providing accurate and stable results.

A final comparison between the methods used showed that Newton-Raphson performed the most in estimation accuracy (lower bias, with lower MSE), especially at large simple sizes. MLE method comes after Newton-Raphson in terms of accuracy and effectiveness in estimation. Among the optimization algorithms, CG was bust but less efficient compared to Newton-Raphson and the best sample size in 200. Therefore, its preferable to use Newton-Raphson when large samples are a variable to achieve accurate and reliable estimates. As for small sample, MLE or CG can be relied upon as effective alternative options.

Application

The partical aspect represents an important part in confirming the efficiency and flexibility of the proposed distribution. Therefore, real neutrosophic data (containing uncertain values) were used, represented by mortality rate for children under five years [37]. The results of NGoIWD were also compared with six other new distributions that were employed within N.L, namely [N.Beta inverse Weibull (NBeIWD) (New), N.Kumaraswamy inverse Weibull (NKuIWD) (New), N.[0,1]Truncated Exponential exponential inverse Wiebull (NTEEIWD), (New), N.odd Lomax inverse Weibull (NLoIWD) (New), N.[0,1] Nadarajah Haghighi inverse Weibull (N[0,1] NHEIWD) (New), and N. inverse Weibull (NIWD) (New)], the comparison was conducted using statistical measures, AIC, BIC, HQIC, and CAIC [38], [39], in addition to Kolmogorov-Smirnov statistic (KS) and the Anderson-Darling statistic (A), Cramér-von Mises statistic (W), and p-value, in addition to estimating the parameters for each distribution, which are show in Tables 6, 7, and 8, respectively.

Table 6: Value of log-likelihoods, and some criteria for data

Dist.	-2L	AIC	CAIC	BIC	HQIC
NGoIWD	[85.08047,85.14456]	[178.1609,178.2891]	[180.0657,180.1939]	[183.1933,183.3215]	[179.6101,179.7383]
NBeIWD	[85.6218,85.82657]	[179.2436,179.6531]	[181.1484,181.5579]	[184.276,184.6855]	[180.6927,181.1023]
NKuIWD	[85.68361,85.88461]	[179.3672,179.7692]	[181.272,181.674]	[184.3996,184.8016]	[180.8164,181.2184]
NTEEIWD	[85.77587,85.96429]	[179.5517,179.9286]	[181.4565,181.8333]	[184.5841,184.961]	[181.0009,181.3777]
NLoIWD	[85.78574,85.98495]	[179.5715,179.9699]	[181.4762,181.8747]	[184.6039,185.0023]	[181.0206,181.419]
N[0,1] NHIWD	[85.55605,85.70042]	[179.1121,179.4008]	[181.0169,181.3056]	[184.1445,184.4332]	[180.5612,180.85]
NIWD	[87.57488, 89.04236]	[179.1498, 182.0847]	[179.6715, 182.6065]	[181.666, 184.6009]	[179.8743, 182.8093]

Table 6 compares different distributions using some criteria. NGoIWD showed superior performance in most of measures, which reinforces its importance as a better model of probability distribution in studied data.

Table 7: Value of statistical measures for data

Dist.	W	A	K-S	p-value
NGoIWD	[0.03698845,0.04198438]	[0.3053922,0.3124517]	[0.09656345,0.109591]	[0.8803729, 0.94937]
NBeIWD	[0.04135859,0.05307781]	[0.3352919,0.3749891]	[0.1005925, 0.1083155]	[0.8884535,0.9313852]
NKuIWD	[0.04118346,0.05254187]	[0.3357333,0.3739378]	[0.1053122, 0.1063571]	[0.9003333,0.9064056]
NTEEIWD	[0.04308012,0.05133014]	[0.3468346,0.3718426]	[0.105257,0.109166]	[0.883098, 0.9067212]
NLoIWD	[0.04137385,0.05248551]	[0.3762638,0.350955]	[0.1070826,0.1182845]	[0.8960071, 0.905707]
N[0,1] NHIWD	[0.03991631,0.04809916]	[0.3280546,0.3388238]	[0.1034424,0.105434]	[0.8190817, 0.916790]
NIWD	[0.05156488,0.06693629]	[0.3897633,0.4487718]	[0.1293297, 0.163349]	[0.4447074,0.7295865]

Table 7 shows the statistics of proportional reassures, where NGoIWD shows a high fit with data compared to other distributions, which highlights its effectiveness in testing model fit.

Table 8: Value of parameter estimators by MLE for data

Dist.	$\hat{r}_N$	$\hat{u}_N$	$\hat{q}_N$	$\hat{p}_N$
NGoIWD	[1.2333345, 1.6759653]	[0.3070585, 0.4011926]	[37.8950656, 50.0533534]	[1.3250379, 1.3270323]
NBeIWD	[13.2524126, 13.875188]	[14.152517,16.3963211]	[3.4613962, 4.059859]	[0.5523993, 0.646267]
NKuIWD	[6.5300683, 9.702639]	[7.020073,8.3708336]	[4.7151010, 4.759534]	[0.9445675, 1.093304]
NTEEIWD	[36.8596699, 39.9847668]	[0.7559911, 5.4482379]	[8.8413262,27.0159664]	[0.4069596,0.6904264]
NLoIWD	[2.0224036, 2.9191380]	[25.0074481, 27.8838125]	[12.0188809, 17.2014241]	[0.7846065, 0.9245617]

N[0,1] NHIWD	[10.2879695, 11.6154147]	[0.3563667,0.4302104]	[25.6349516, 35.6783356]	[0.9312000, 1.0386221]
NIWD	---	---	[88.913221,99.403537]	[1.795396,1.905606]

Table 8, presents parameter estimates using the MLE method for different distributions, and shows the superiority of NGoIWD in providing accurate estimates for a range of parameters compared to other distributions.

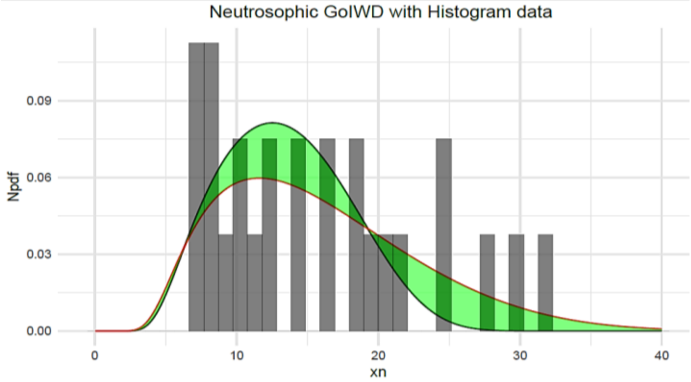


Figure 7: Fitting pdfs NGoIWD with histogram data set

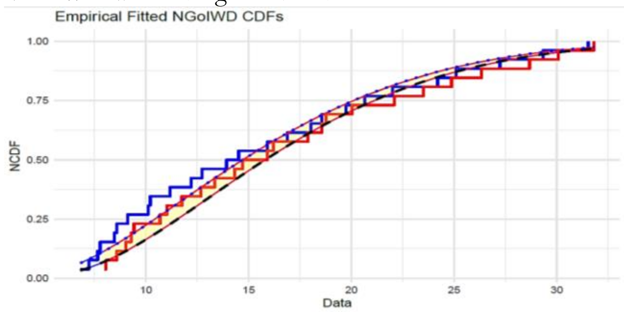


Figure 8: Empirical Fitted CDFs NGoIWD with data set

The results in figure 7 shoe a high agreement between theoretical NGoIWD and the actual data, as the figure highlights the model's to accurately represent data. This is reflected in the close alignment between the observed values and the predicted values across different levels of probability.

Form figure 8, the theoretical CDF matches closely with the actual data, which enhances the model's ability to represent the cumulative behavior of the data. The distribution lines show a match across all probability levels, which gives greater confidence in using NGoIWD to analyze data containing uncertain values.

CONCLUSIONS

Based on the analysis of results of the study, we conclude the following:

1. The proposed NGoIWD has proven its efficiency in dealing with uncertain data compared to other distributions such as NBeIWD, NKuIWD, and NIWD. The NGoIWD showed high flexibility in estimation and achieved great agreement with the actual data, as evidenced by the results of statistical criteria such as AIC, BIC, and HQIC.
2. The results shown by (figure 7, and 8) confirmed that NGoIWD is able to accurately represent that actual data, whether through the N.pdf or N.CDF.
3. The Newton-Raphson method showed excellent performance in estimating parameters, as it provided the lowest bias and lowest MSE compared to other estimation methods, especially when using large samples 200.
4. When applied to real data, the NGoIWD outperforms other distributions, demonstrating its efficiency in analyzing neutrosophic data.
5. The NGoIWD provides flexible extensions such as general functions, moments, and CDF, enhancing its use in a variety of statistical and applied fields.

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